4. ARSENIN V. Ya., *Methods of Mathematical Physics and Special Functions*. Nauka, Moscow, 1974.
5. GRINBERG G. A., On a possible method of approaching the consideration of problems of heat conduction, diffusion, wave theory and the like in the presence of moving boundaries, and some of its applications. *Prikl. Mat. Mekh.* **31**, 2, 193–203, 1967.
6. TYULIN V. N., *Introduction to the Theory of Radiation and Scattering of Sound*. Nauka, Moscow, 1975.

Translated by D.L.

J. Appl. Maths Mechs Vol. 55, No. 6, pp. 948–953, 1991
Printed in Great Britain.
0021–8928/91 \$15.00+.00

A SOLUTION OF AN INVERSE PROBLEM FOR THE WAVE EQUATION WITH NON-LINEAR CONDITIONS IN DOMAINS WITH MOVING BOUNDARIES†

V. S. KRUTIKOV

Nikolayev

(*Received* 22 *March* 1990)

A new approach is proposed for solving problems with moving boundaries. Assuming spherical symmetry, wave phenomena are considered in the case of a surface, of arbitrary initial radius, moving in a compressible medium at a velocity governed by an arbitrary law. Formulas suitable for solving both the inverse and direct problems are obtained.

ATTEMPTS to allow for the mobility of the boundaries in wave-equation situations have hitherto been confined mainly to cases in which the boundary conditions are satisfied on the moving boundaries (the direct problem) [1, 2]. The method used in [1] reduces such situations to an infinite system of first-order linear differential equations. In the case considered below an additional condition is specified not at the moving boundary but at a fixed point of the wave zone (the inverse problem), and the problem is to determine the functions of interest at other points, including the moving boundaries. This is to be done without knowledge of the law governing the variation of the boundaries, which is also to be determined. In addition, the additional condition may even be non-linear.

The essence of the approach is to determine the relationship between the values of the unknown functions at the moving boundaries and at other points taking into account the actual delays [3]. In some cases, such as a moving cylindrical surface [4] or a penetrable spherical boundary [5], explicit formulas can be derived for the functions.

†*Prikl. Mat. Mekh.* Vol. 55, No. 6, pp. 1059–1062, 1991.

1. LINEAR ADDITIONAL CONDITIONS

Consider the problem

$$\varphi_{tt} - a^2\varphi_{rr} - 2a^2r^{-1}\varphi_r = 0, \quad r \geqslant R(t) \tag{1.1}$$

$$\varphi_r(r, 0) = \varphi(r, 0) = 0, \quad R(0) = r_0 \tag{1.2}$$

$$\rho\varphi_t\,|_{r=r_1} = P = f(t - (r_1 - r_0)/a) \tag{1.3}$$

$$-\varphi_r\,|_{r=R(t)} = v \tag{1.4}$$

where t is the time, $R(t)$ and r_0, r, r_1 are the coordinates of the moving boundaries, the initial and current points of the wave zone, respectively, and a and ρ are constants. If conditions (1.3) are known and we have to reconstruct the values of the unknown function φ and its derivatives at any other points, including the moving boundaries, this is the inverse problem; if only conditions (1.4) are known, this is the direct problem.

Evaluating the one-sided Laplace transform to the wave equation (1.1), taking the homogeneous initial data into account, we obtain an operator equation

$$\bar{\varphi}_{rr}(r, s) + 2r^{-1}\bar{\varphi}_r(r, s) - s^2a^{-2}\bar{\varphi}(r, s) = 0$$

whose solution is

$$\bar{\varphi}(r, s) = r^{-1}[c_1(s)\exp(-sr/a) + c_2(s)\exp(sr/a)]$$

where s is the transform parameter.

Let us assume that the boundary is moving in a medium with no boundaries; in accordance with condition (1.3), we define $c(s) = f(s)r_1(s\rho)^{-1}\exp(sr_0/a)$. Then the solution of the wave equation can be written as

$$\bar{P}(r, s) = \rho[s\bar{\varphi} - \varphi(r, 0)], \quad \bar{v}(\bar{r}, s) = \left(s\frac{r}{a} + 1\right)\frac{\bar{\varphi}(r, s)}{r}$$

$$\bar{\varphi}(r, s) = \frac{r_1}{r\rho}\frac{\bar{f}(s)}{s}\exp\left(-s\frac{r - r_0}{a}\right)$$

Returning to source functions, we obtain the required functions taking the real-time delays into account. The values of the functions at any points are

$$P(r, t) = \frac{r_1}{r}f(\xi), \quad v(r, t)\frac{r^2\rho}{r_1} = \frac{r}{a}f(\xi) + \int_0^t f(\xi)\,dt,$$

$$\varphi(r, t) = \frac{r_1}{r\rho}\int_0^t f(\xi)\,dt, \quad \xi = t - \frac{r - r_0}{a} \tag{1.5}$$

At the moving boundary

$$P(R(t), t) = \frac{r_1}{R(t)}f\left(t - \frac{R(t) - r_0}{a}\right), \quad v(R(t), t)\frac{R^2(t)\rho}{r_1} =$$

$$= \frac{R(t)}{a}f\left(t - \frac{R(t) - r_0}{a}\right) + \left[\int_0^t f(\xi)\,dt\right]_{r=R(t)},$$

$$\varphi(R(t), t) = \frac{r_1}{R(t)\rho}\left[\int_0^t f(\xi)\,dt\right]_{r=R(t)} \tag{1.6}$$

Calculations with formulas (1.6), when inverse problems are being dealt with, require a knowledge of $R(t)$. The variation of the radius of the moving boundary may be determined as follows. It is known [6] that the volume of a liquid flowing through a closed surface ($4\pi r^2$) is equal to the change in volume per unit time, i.e. $dV/dt = 4\pi r^2 v(r, t)$, where v is the second function in (1.5). Integrating from 0 to t and transferring to the moving boundary, we obtain a cubic equation:

$$\frac{[R^3(t) - r_0^3]\rho}{3r_1} = \left[\int_0^t \frac{r}{a} f(\xi)\,dt + \int_0^t\int_0^t f(\xi)\,dt\,dt\right]_{r=R(t)} \tag{1.7}$$

In inverse problems the function f in (1.7) is known.

An arbitrary function f can be approximated in various ways [3]. Let

$$P(r_1, t) = Ae^{-\alpha\xi_1}\sigma_0(\xi_1), \quad A, \alpha = \text{const}, \quad \xi_1 = t - \frac{r_1 - r_0}{a}, \quad \sigma_0(t-\tau) = \begin{cases} 0, & t < \tau \\ 1, & t \geqslant \tau \end{cases}$$

(σ_0 is the unit discontinuous function of zero order). Then in view of (1.5)–(1.7) we obtain the formulas

$$P(r, t) = \frac{r_1}{r} Ae^{-\alpha\xi}\sigma_0(\xi), \quad v(r, t) = \frac{r_1 A}{\rho r^2}\left\{\frac{1}{\alpha} + \left(\frac{r}{a} - \frac{1}{\alpha}\right)e^{-\alpha\xi}\right\} \tag{1.8}$$

$$P(R(t), t) = \frac{r_1}{R(t)} Ae^{-\alpha\xi}\sigma_0(\xi), \quad v(R(t), t) = \frac{r_1 A}{\rho R^2(t)}\left\{\frac{1}{\alpha} + \left(\frac{R(t)}{a} - \frac{1}{\alpha}\right)e^{-\alpha\xi}\right\} \tag{1.9}$$

$$\frac{[R^3(t) - r_0^3]\rho}{3r_1} = \frac{R(t)}{a}\frac{A}{\alpha}(1 - e^{-\alpha\xi}) + \frac{A}{\alpha}\left[\xi - \frac{1}{\alpha}(1 - e^{-\alpha\xi})\right] \tag{1.10}$$

In the general case of boundaries whose motion is governed by arbitrary laws and an arbitrary function f, the latter is conveniently approximated by a Lagrange polynomial of degree m:

$$f = \sum A_m \xi_1^m, \quad A_m = \text{const} \tag{1.11}$$

Throughout this note, unless otherwise stated, summation is performed over m from 0 to infinity. The number m of interpolation points (exact values of the function) may be as large as desired [7].

The sequence of calculations in solving inverse problems is as follows. Use formulas (1.7) and (1.10) to determine the change in the radius of the moving boundary, and formulas (1.6) and (1.9) to determine the values of the unknown functions on the boundary. The values of these functions at other points may then be determined by using (1.5) and (1.8).

The solutions thus computed for the wave equation with moving boundaries, i.e. formulas (1.1)–(1.4), may be used to describe the expansion of a sphere in a compressible medium. Here the function $P(R(t), t)$ has to be computed taking into account the non-linear term of the Cauchy–Lagrange integral $0.5\rho\varphi_r^2$ [8]. Substitution of the solutions (1.5), (1.6), (1.8) and (1.9) into the wave equation makes its left-hand side vanish; as $a \to \infty$, these formulas reduce to known solutions for an incompressible medium. Indeed, differentiating Eq. (1.7) twice and letting $a \to \infty$, we obtain the well-known formula $P - P_0 = \rho r^{-1}(2R\dot{R}^2 + R^2\ddot{R})$ for the point r_1. They can be used to solve the direct and inverse problems.

As to the cubic equation (1.7) and the similar equation (1.10) [and the third equation of (2.3) below], these can be solved, e.g. by successive approximations [3]. A good choice for the first approximation is the value computed as $a \to \infty$ or for the preceding instant of time. The formulas possess favourable convergence properties and so one can compute $R(t)$ to any desired accuracy.

2. NON-LINEAR ADDITIONAL CONDITION

Consider the condition

$$P(r,t) = -\rho\left(\varphi_t + \frac{1}{2}\varphi_r^2\right)\Big|_{r=r_1} \tag{2.1}$$

Using expansion (1.11) with conditions (1.5)–(1.7), we can write

$$P(r,t) = \frac{r_1}{r}\sum A_m \xi^m, \quad P(R(t),t) = \frac{r_1}{R(t)}\sum A_m \xi^m \tag{2.2}$$

$$\begin{gathered}\frac{v(r,t)\, r^2 \rho}{r_1} = \frac{r}{a}\sum A_m \xi^m + \sum \frac{A_m}{m+1}\xi^{m+1} \\ \frac{v(R(t),t)\, R^2(t)\rho}{r_1} = \frac{R(t)}{a}\sum A_m \xi^m + \sum \frac{A_m}{m+1}\xi^{m+1} \\ \frac{[R^3(t) - r_0^3]\rho}{3r_1} = \frac{R(t)}{a}\sum \frac{A_m}{m+1}\xi^{m+1} + \sum \frac{A_m}{(m+1)(m+2)}\xi^{m+2} \\ \xi = t - \frac{R(t) - r_0}{a}\end{gathered} \tag{2.3}$$

We will write condition (2.1) in the form

$$P(r,t) = \frac{r_1}{r}\sum A_m \xi^m - \frac{1}{2}\rho\left\{\frac{r_1}{r^2\rho}\left[\frac{r}{a}\sum A_m \xi^m + \sum \frac{A_m}{m+1}\xi^{m+1}\right]\right\}^2 \tag{2.4}$$

The solution of the problem may be reduced to solving a single algebraic equation. We will compute A_0 when the solution, approximated by the Lagrange polynomial $A_0\sigma(\xi_1)$, is a straight line at the point $r = r_1$ for time t_0. This is the time of arrival of the wave at the point r_1. From (2.4) we obtain a quadratic equation for A_0, the solution of which gives

$$A_0 = \frac{1}{2c_1} \pm \sqrt{\left(\frac{1}{2c_1}\right)^2 - \frac{P(r_1,t)}{c_1}}, \quad c_1 = \frac{1}{2\rho a^2}, \quad t_0 = \frac{r_1 - r_0}{a}, \quad \xi = 0 \tag{2.5}$$

We enlarge the time interval by a certain quantity Δt and find the solution, approximated by a Lagrange polynomial $A_0 + A_1\xi$, which passes through t_0 and a second point $t_1 = t_0 + \Delta t$. Then, by (2.1),

$$P(r_1,t_1) = A_0 + A_1\xi - c_2\left\{\left[\frac{r_1}{a}(A_0 + A_1\xi) + A_0\xi + \frac{A_1}{2}\xi^2\right]\right\}^2, \quad c_2 = \frac{1}{2r_1^2\rho} \tag{2.6}$$

The quantity P is known by assumption and A_0 is known from (2.5). Thus A_1 is found by solving the quadratic equation (2.6).

Continuing in this way, we can find the necessary number of coefficients of the Lagrange polynomial. The formula for the mth coefficient will be

$$\begin{gathered}A_m = -\frac{1}{2}\frac{(2c_2c_3c_4 - \xi_1^m)}{c_2c_4^2} \pm \left\{\left[\frac{1}{2}\frac{(2c_2c_3c_4 - \xi_1^m)}{c_2c_4^2}\right]^2 + \frac{c_0 - c_2c_2^2}{c_2c_4^2}\right\}^{1/2} \\ c_0 = -P(r_1,t) + A_0 + A_1\xi_1 + \ldots + A_{m-1}\xi_1^{m-1}, \quad c_2 = \frac{1}{2r_1^2\rho}, \quad \xi_1 = t - \frac{r_1 - r_0}{a} \\ c_3 = \frac{r_1}{a}\sum_{m=0}^{m-1} A_m\xi^m + \sum_{m=0}^{m-1}\frac{A_m}{m+1}\xi_1^{m+1}, \quad c_4 = \frac{r_1}{a}\xi_1^m + \frac{1}{m+1}\xi_1^{m+1}\end{gathered} \tag{2.7}$$

As we see, the solution of a problem with a non-linear additional condition (2.1), where the boundary is moving at an arbitrary velocity, has been reduced to computations with formulas of type (2.7). Knowing the

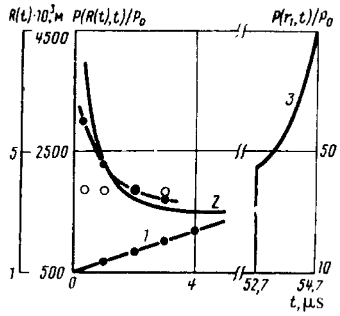

FIG. 1.

coefficients A_m, we can then use (2.2), (2.3) to determine the unknown functions at the moving boundaries and at any other points, as well as the parameters of the motion of the boundary.

The solutions of the inverse problem for the wave equation with non-linear conditions (2.1) in domains with moving boundaries may be used, e.g. to solve problems that arise when the behaviour of expanding cavities in compressible media has to be controlled [3].

3. EXAMPLE

Consider the problem in Sec. 2. Figure 1 shows the results of computations, carried out by the method of characteristics [3, 9], for the system of equations of motion, continuity and state for iso-entropic processes in Tait's form:

$$v_t - vv_r + \rho^{-1}P_r = 0, \quad \rho_t + (\rho v)_r' + (\nu - 1)\rho v = 0, \quad (P + B)/(P_0 + B) = (\rho/\rho_0)^n \tag{3.1}$$

with the appropriate boundary and initial conditions (B and n are constants and ν is the symmetry exponent). The law of expansion of the sphere is $v(R(t), t) = 350\exp(-10^3 t)$, $r_0 = 10^{-3}$ m, the radius $R(t)$ and the pressure at the moving boundary and at the point $r_1 = 0.08$ m in the wave zone are represented by curves 1, 2, 3, respectively.

Using formula (2.7) and successively determining the coefficients of the Lagrange polynomial, we first determine A_0, relying on the known values of $P(r_1, t)$ (curve 3) at $\rho = 102$ kg s^2/m^4, $a = 1500$ m/s, $t = 52.7$–55.7×10^{-6} s, and then go on to find further coefficients A_m: $A_0 = 49.46$; $A_1 = 3.56 \times 10^4$; $A_2 = 3.46 \times 10^{12}$; $A_3 = -6.355 \times 10^{16}$.

The other unknown functions are now determined using (2.2) and (2.3). The computed values of $R(t)$ and $P(R(t), t)$ are represented by solid circles and the values computed by the known formula $P - P_0 = \rho(3/2\, \dot{R}^2 + R\ddot{R})$ by open circles. Such results cannot be obtained by other methods; the method used in [1] is not easily applicable to inverse problems.

The satisfactory agreement between the results of different computations indicates that the approach proposed above is indeed capable of dealing with this type of problem.

REFERENCES

1. GRINBERG G. A., On a possible method of approaching the consideration of problems of heat conduction, diffusion, wave theory and the like in the presence of moving boundaries, and some of its applications. *Prikl. Mat. Mekh.* **31**, 2, 193–203, 1967.
2. TUCHKEVICH V. M. (Ed.), *Problems in Mathematical Physics*. Nauka, Leningrad, 1976.

3. KRUTIKOV V. S., *One-dimensional Problems of the Mechanics of a Continuous Medium with Moving Boundaries*. Naukova Dumka, Kiev, 1985.
4. KRUTIKOV V. S., On the determination of the pressure on the moving boundary of a plasma piston. *Pis'ma v KhTF* **14**, 6, 510–514, 1988.
5. KRUTIKOV V. S., Approximate evaluation of the effect of the penetrability of the moving boundary of a plasma piston. *Pis'ma v ZhTF* **15**, 14, 45–48, 1989.
6. LANDAU L. D. and LIFSHITS E. M., *Mechanics of Continuous Media*. Gostekhizdat, Moscow, 1954.
7. VINOGRADOV I. M. (Ed.), *Mathematical Encyclopaedia* (Sovetskaya Entsiklopedia) Vol. 2. Moscow, 1979.
8. SLEPYAN L. I., On the equations of the dynamics of an axially symmetric cavity in an ideal compressible liquid. *Dokl. Akad. Nauk SSSR* **282**, 4, 809–813, 1985.
9. ROZHDESTVENSKII B. L. and YANENKO N. N., *Systems of Quasilinear Equations and their Applications to Gas Dynamics*. Nauka, Moscow, 1978.

Translated by D.L.

J. Appl. Maths Mechs Vol. 55, No. 6, pp. 953–957, 1991
Printed in Great Britain.
0021–8928/91 $15.00+.00

A NON-AXISYMMETRIC CONTACT PROBLEM IN THE CASE OF A NORMAL LOAD APPLIED OUTSIDE THE AREA OF CONTACT†

V. I. MOSSAKOVSKI and YE. V. POSHIVALOVA

Dnepropetrovsk

(*Received* 11 *July* 1990)

1. A FORMULA describing the effect of a load acting outside a circular stamp in a plane is known [1]. Below we propose a novel approach to the study of the pressure under a non-axisymmetric plane stamp when normal forces are applied to the free surface of an elastic half-space. The approach includes the method, proposed by Mossakovskii, of reducing the three-dimensional problem of potential theory to a plane problem. The main merit of this method, as compared with that in [2] based on the Sommerfeld method, is the possibility of constructing effective numerical algorithms, since any subsequent approximation can be constructed indepndently of the preceding one, by adding some supplementary terms. The problem in question is reduced, in the final analysis, to a system of plane problems of potential theory whose boundary conditions contain trigonometric polynomials with unknown coefficients, which can be determined from the condition that the solution is regular within the area of contact.

Let a normal force R be applied to the surface of an elastic half-space outside the area of contact at the point ξ, η. As a result, additional pressure and normal displacement occur under the stamp.

We will assume that the normal displacement of the stamp $W(\rho, \alpha)$ is identical with the displacement of the

†*Prikl. Mat. Mekh.* Vol. 55, No. 6, pp. 1062–1065, 1991.